AN INVESTIGATION OF OPTIMAL DECISION RULES FOR SEVERAL SINGLE-PERIOD STOCHASTIC INVENTORY PROBLEMS

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THESIS

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September 1973

An Investigation of Optimal Decision Rules for Several Single-Period Stochastic Inventory Problems

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Submitted in partial fulfillment of the
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MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1973

ABSTRACT

The single-period inventory model, known as the newsboy or Christmas tree problem, is extended for several cost expressions and optimal decision rules for these variations are derived. A two-echelon single-period inventory model is developed and optimal decision rules are found. Conclusions and suggested extensions for the newsboy problem and of the two-echelon inventory model are discussed.

TABLE OF CONTENTS

I. SUMMARY

The purpose of the study reported in this thesis was to examine some variations on the standard single-period inventory model and to extend this standard model to a two-echelon single-period inventory model. The single-period inventory model is well known as a newsboy problem or a Christmas tree problem, since it can be phrased in terms of deciding how many trees a dealer in Christmas trees should purchase for the season, or how many newspapers a newsboy should buy on a given day for his customers.

One modification to the standard single-period model is to use a quadratic cost function where the cost of stockout or shortage items will be proportional to the square of the amount short or left over. With the modification it was possible to derive optimal decision rules in the sense of minimizing expected cost. These rules, however, are somewhat intractable.

Another extension is the fixed shortage cost case where any number of items short will cost K, a fixed cost, to the system. In this case a numerically tractable decision rule was derived which permits computation of optimal order quantities for a large variety of demand distributions.

A newsboy problem is concerned with only one element in the inventory system, the newsboy himself. A two-echelon model is structured with several newsboys and one distributor

as a system. In case of stockout each newsboy may be able to get a special delivery from the distributor, with extra cost to the newsboy. The decisions that must be made are the number of newspapers that should be ordered by the distributor, and the number of newspapers that should be stocked by each newsboy. A useful two-stage decision rule is found for this case which yields optimal order quantities for both the distributor and the newsboys.

II. INTRODUCTION

The single-period inventory model which is sometimes called the newsboy problem or the Christmas tree problem is very well known in the field of management decision making, especially in inventory management. This paper is devoted to variations and extensions of this newsboy problem.

The purpose of this paper is to consider the newsboy problem or the single-period inventory model in various ways. Different problems have their own cost functions which lead to different decision rules. The objective here is to extend the classic newsboy problem for both a variety of cost functions and for a two-echelon inventory system which is composed of several newsboys and a distributor, considered as one inventory system.

The study will only consider single period models with time independent cost. This excludes costs which are proportional to the length of time that a unit remains in inventory, or a stockout cost which is proportional to the length of time from when the demand occurs until the end of the period when stocks are replenished.

Chapter III reviews the standard model of the newsboy problem. Chapter IV is devoted to some extensions and applications of this standard model. Some different cost functions will be used and various decision rules are derived. Chaper V is concerned with a two-echelon inventory system

which may be viewed as a distributor with several newsboys. The problem is to minimize expected cost to the system. The concluding chapter proposes several extensions of the two-echelon inventory model.

III. STANDARD SINGLE-PERIOD INVENTORY MODELS

The general single-period inventory model is known as the Christmas tree problem or the newsboy problem, since it can be phrased in terms of deciding how many trees a dealer in Christmas trees should purchase for the season, or how many newspapers a boy should buy on a given day for his corner newsstand.

The essential characteristic of the model is that only a single time period, usually a finite length, is relevant and only a single procurement is made in each period. Stock-outs can not be refilled and the items left at the end of the period can not be transferred for use in the next period. The overage items may be thrown away or sold at a bargain price. The model provides a representation of the trade off between shortage and surplus costs.

In this chapter we shall introduce notation for the newsboy problem and give well-known results of the standard single period model for readers who may not be familiar with it.

The general single period model with time independent cost may be illustrated by considering the newboy's problem. He has to order the newspaper one day in advance for the next day's sale. His problem is to determine his order quantity. He has only one chance to place an order for each day, with no reordering or returning the newspapers.

We wish to construct a decision rule for the newsboy such that the expected cost for the period is minimized. If I_o is the optimum order quantity, or initial inventory level, then the expected cost with (I_o+1) items or (I_o-1) items will be greater than the expected cost with I_o.

Let

I = Inventory level at the beginning of the period,

I_o = Optimum inventory level at the beginning of a time period,

D = Customer demand for the period,

c_1 = Surplus cost per item for items left at the end of the period,

c_2 = Shortage cost per item for items short at the end of the period,

$p(D)$ = Probability that demand is D units, where D is a discrete variable and the distribution p(D) is assumed to be known,

and

$E(I)$ = Expected relevant cost for the period.

The cost equation is

$$\text{Cost} = \begin{cases} c_1(I - D), & D = 0,1,2,\ldots,I \\ c_2(D - I), & D = I+1, I+2, \ldots, \end{cases}$$

and the expected cost for I items, E(I), is

$$E(I) = c_1 \sum_{D=0}^{I} (I-D)p(D) + c_2 \sum_{D=I+1}^{\infty} (D-I)p(D).$$

An optimum initial inventory level, I_o, is obtained by minimizing E(I). This is done by working with two necessary conditions for a minimum;

$$E(I_o) < E(I_o-1) ,$$

$$E(I_o) < E(I_o+1) .$$

Using the expected cost equation, we obtain

$$E(I-1) = E(I) - (c_1+c_2)\ P(I-1) + c_2,$$

and

$$E(I+1) = E(I) + (c_1+c_2)\ P(I) + c_2,$$

where $P(I-1) = \sum_{I=0}^{I-1} p(I)$ is the distribution function.

Then we obtain the decision rule[1]

$$P(I_o-1) < \frac{c_2}{c_1 + c_2} < P(I_o). \qquad (1)$$

The value of the inventory level, I, which minimizes the expected cost, E(I), is that optimum value I_o which

satisfies the above inequality condition. The newsboy should order I_o newspapers to minimize his expected total cost.

If demand is assumed to be a continuous random variable, then the expected cost per period with order quantity I is

$$E(I) = c_1 \int_{D=0}^{I} (I-D)f(D)\, dD + c_2 \int_{D=I}^{\infty} (D-I)f(D)\, d(D).$$

Differentiating E(I) with respect to I and setting the result equal to zero yields

$$F(I_o) = \frac{c_2}{c_1 + c_2}, \qquad (2)$$

where $F(I) = \int_{D=0}^{I} f(D)\, dD.$

Since $\left.\frac{d^2E(I)}{dI^2}\right|_{I=I_o} \geq 0$ then the total expected cost is minimized for that value of I which satisfies the above condition.

The next chapter investigates two variations on this standard newsboy problem.

IV. TWO VARIATIONS ON THE SINGLE-PERIOD MODEL

Although there are many real world problems that fit the characteristics of the standard single-period model, there are some problems which have requiring modification of the standard model. This chapter will be devoted to studying two variations on the standard single-period model.

A. QUADRATIC COST FUNCTIONS

One investigating extension of the newsboy problem is to let surplus or shortage costs vary as the square of the quantities short or surplus. With quadratic functions, we have

$$\text{cost} = \begin{cases} c_1(I-D)^2 , & D = 0,1,\ldots,I \\ c_2(D-I)^2 , & D = I+1,\ldots,D_{max}. \end{cases}$$

We wish to find decision rules which minimize expected costs for the quadratic cost functions, and we will consider both the discrete and continuous cases.

When demand is discrete, the expected cost for I items, E(I), is

$$E(I) = \sum_{D=0}^{I} c_1(I-D)^2\, p(D) + \sum_{D=I+1}^{\infty} c_2(D-I)^2\, p(D).$$

First order difference condition for a minimum at I_o yields

$$-(c_1-c_2)P(I_o)+c_2 < 2[\sum_{D=0}^{I_o} c_1(I_o-D)p(D) - \sum_{D=I_o+1}^{\infty} c_2(D-I_o)p(d)]$$

$$< (c_1-c_2)P(I_o-1) + c_2. \qquad (3)$$

When demand is continuous, the expected cost is

$$E(I) = \int_{D=0}^{I} c_1(I-D)^2 f(D)dD + c_2 \int_{D=I}^{\infty} (D-I)^2 f(D)\, dD.$$

The optimum condition for I in this case is

$$F(I_o) = \frac{c_1 + c_2}{I_o\,(c_1 - c_2)} \int_{D=0}^{I_o} Df(D)dD - \frac{c_2}{I_o\,(c_1-c_2)} (I_o-E(D)). \qquad (4)$$

Derivations for the optimal conditions for I are given in Appendix A.

These decision rules are somewhat intractible, although in some cases it may be possible to solve for I_o numerically.

B. FIXED SHORTAGE COST CASE

In this case a constant cost will be incurred when the inventory has a stockout, regardless of the number of items short. This kind of cost function might arise in such problems as determining the number of bombs for bombers to carry in one flying mission. An excess number of bombs will cost the Air Force c_1 dollars each. A shortage of bombs, any

number at all, will cause the mission to be incomplete, this will cost the Air Force K dollars.

A similar example is the number of shells for artillery company to take with them from the ammunition dump. An excess number will cost the company c_1 dollars each. The shortage of shells, any number at all, will cost the company commander to send a truck back to the ammunition dump to get them, which will cost the company K dollars.

The cost function here is

$$\text{cost} = \begin{cases} c_1(I-D) & , \quad D = 0,1,\ldots,I \\ K & , \quad D = I+1,\ldots,D_{max}, \end{cases}$$

where K is the constant cost.

When demand is discrete and $p(D) > 0$, $D = 0,1,\ldots,D_{max}$, the optimum condition for I is

$$\frac{P(I_o-1)}{p(I_o)} < \frac{K}{c_1} < \frac{P(I_o)}{p(I_o+1)} \quad . \tag{5}$$

When demand is continuous, the optimum condition for I is

$$F(I_o) = \frac{K}{c_1} f(I_o) \quad . \tag{6}$$

Derivations of these results are contained in Appendix B. The expected cost is minimized for that optimum value of I, I_o, which satisfies the above conditions.

V. TWO-ECHELON SINGLE PERIOD INVENTORY MODEL

A distributor-retailer system may be viewed as a two-echelon single period inventory model. Instead of considering the newsboy or the retailer alone, we put several retailers and a distributor together and allow the retailers to make a special order in case of stockout. The interchange of items among the retailers is assumed to have no cost at all. The decision variables are the amount to order by the distributor and the amount to order by each retailer. In this case we are trying to minimize the total of costs to this distributor-retailer system. (See Figure I.)

Let

I = Initial inventory level at distributor,

J = Inventory level at retailers,

n = Number of retailers in the system,

$I+nJ$ = Total order by system,

D_i = Demand at each retailer, $i = 1,2,\ldots,n$,

c_1 = Surplus cost per unit of goods at retailers,

c_2 = Retailer special order cost per unit of goods, (Immediately delivered units from the distributor in case of stockout at reatilers),

c_3 = Surplus cost per unit of goods at distributor,

c_4 = Shortage cost per unit of goods at distributor,

$p(D)$ = Probability density function of demand of each retailer, i.e., $p(D_1)=p(D_2)=\ldots=p(D)$,

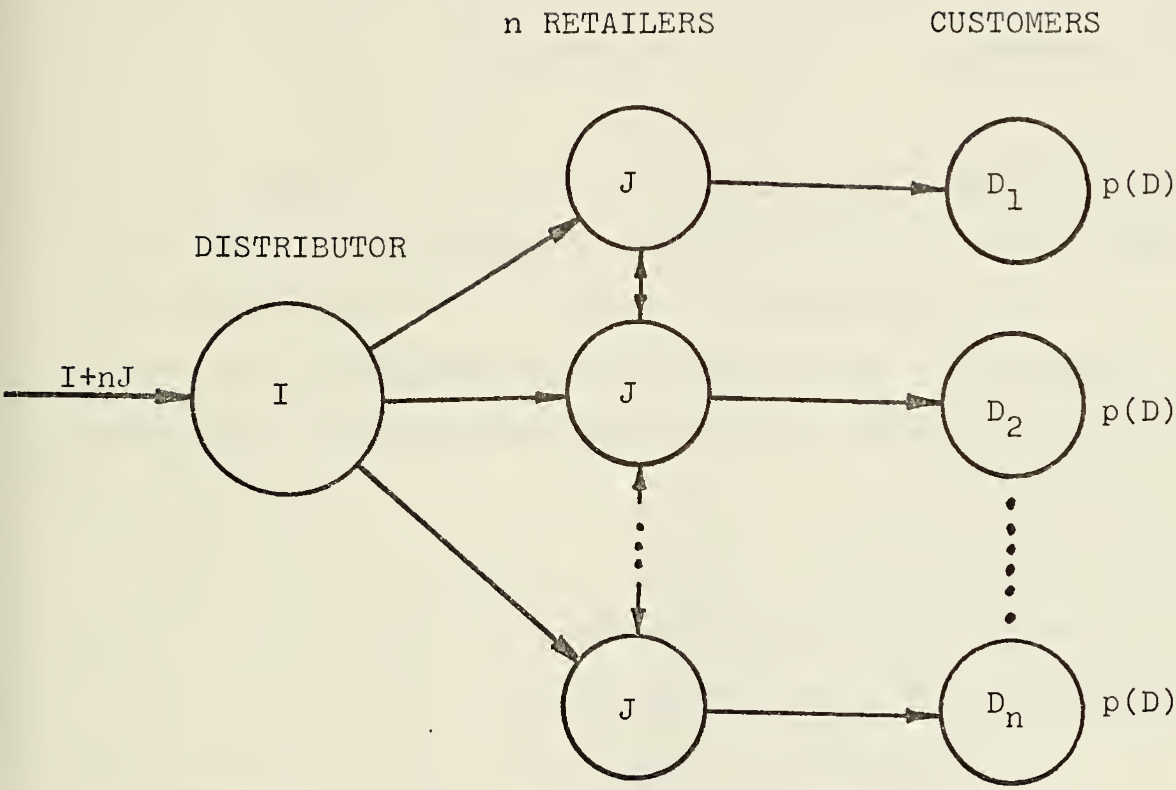

FIGURE I . Two-Echelon Inventory System.

$P(D)$ = Probability distribution function of demand,

and

$$DD = D_1 + D_2 + \ldots + D_n = \text{Total demand for system.}$$

Then the cost equation to the system is

$$\text{cost to system} = \begin{cases} c_1(nJ-DD) + c_3 I & , DD=0,1,\ldots,nJ \\ c_2(DD-nJ) + c_3(I+nJ-DD), & DD=nJ+1,\ldots,I+nJ \\ c_4(DD-I-nJ) + c_2 I & , DD=I+nJ+1,\ldots. \end{cases}$$

We assume that the system's <u>structure</u> is optimal. Thus since the retail inventory will be exhausted before using the special delivery, it must be true that $c_3 < c_1 + c_2$. Since the system requires that when retail inventory is zero, a special delivery will be used, it must be true that $c_2 < c_3 + c_4$

The problem is how to construct a decision rule to find the optimum amount of I, the inventory level at the distributor, and nJ, the total inventory levels of the retailers, such that the cost to the system is minimized.

The expected cost equation is

$$E(I,nj) = \sum_{DD=0}^{nJ} [c_1(nJ - DD) + c_3 I]\, p(DD)$$

$$+ \sum_{DD=nJ+1}^{I+nJ} [c_2(DD-nJ) + c_3(I+nJ-DD)]\, p(DD)$$

$$+ \sum_{DD=I+nJ+1}^{\infty} [c_4(DD-I-nJ) + c_2 I]\, p(DD).$$

If $E(I,nJ)$ is the minimum expected cost, then these conditions will hold:

$$E(I,nJ) < E(I+1,nJ) \tag{7}$$

$$E(I,nJ) < E(I-1,nJ) \tag{8}$$

$$E(I,nJ) < E(I,nJ+1) \tag{9}$$

$$E(I,nJ) < E(I,nJ-1) \ . \tag{10}$$

From these four conditions, the decision rules may be constructed. The right-hand side of (7) is

$$
\begin{aligned}
E(I+1,nJ) = & \sum_{DD=0}^{nJ} [c_1(nJ-DD) + c_3(I+1)]\, p(DD) \\
& + \sum_{DD=nJ+1}^{I+nJ+1} [c_2(DD-nJ) + c_3(I+1+nJ-DD)]\, p(DD) \\
& + \sum_{DD=I+nJ+2}^{\infty} [c_4(DD-I-1-nJ) + c_2(I+1)]\, p(DD) \\
= & \ E(I,nJ) + c_3P(nJ) + \sum_{DD=nJ+1}^{I+nJ} c_3p(DD) \\
& +[c_2(I+nJ+1-nJ) + c_3(I+1+nJ-(I+1-nJ))]p(I+nJ+1) \\
& + \sum_{DD=I+nJ+1}^{\infty} (c_2-c_4)p(DD) - c_4((I+nJ+1)-(I+nJ+1)) \\
& \qquad p(I+nJ+1) \\
& - c_2(I+1)p(I+nJ+1)
\end{aligned}
$$

$$E(I+1,nJ) = E(I,nJ) + c_3P(nJ) + c_3(P(I+nJ) - P(nJ))$$

$$+ c_2(I+1)p(I+nJ+1) - (c_4-c_2) + (c_4-c_2)P(I+nJ)$$

$$- c_2(I+1)p(I+nJ+1)$$

$$= E(I,nJ) + (c_3+c_4-c_2)P(I+nJ) - (c_4-c_2).$$

From condition (7)

$$E(I+1,nJ) - E(I,nJ) > 0$$

we have

$$E(I+1,nJ) - E(I,nJ) = (c_3+c_4-c_2)P(I+nJ)-(c_4-c_2) > 0,$$

or

$$P(I+nJ) > \frac{c_4 - c_2}{c_3 + c_4 - c_2} \quad . \qquad (11)$$

The right-hand side of (8) is

$$E(I-1,nJ) = \sum_{DD=0}^{nJ} [c_1(nJ-DD) + c_3(I-1)]p(DD)$$

$$+ \sum_{DD=nJ+1}^{I+nJ-1} [c_2(DD-nJ) + c_3(I-1+nJ-DD)]p(DD)$$

$$+ \sum_{DD=I+nJ}^{\infty} [c_4(DD-I+1-nJ) + c_2(I-1)]p(DD)$$

$$= E(I,nJ) - c_3P(nJ) - \sum_{DD=nJ+1}^{I+nJ} c_3p(DD)$$

$$- [c_2I-c_3]p(I+nJ) + \sum_{DD=I+nJ+1}^{\infty} (c_4-c_2)p(DD)$$

$$+ [c_2(I-1)+ c_4]p(I+nJ)$$

$$= E(I,nJ) - c_3P(nJ) - c_3P(I+nJ) + c_3P(nJ)$$

$$+ c_3p(I+nJ) + (c_4-c_2) - c_4P(I+nJ) + c_2P(I+nJ)$$

$$- c_2p(I+nJ) + c_4p(I+nJ)$$

$$= E(I,nJ) - (c_3+c_4-c_2)[P(I+nJ)-p(I+nJ)] + (c_4-c_2)$$

$$= E(I,nJ) - (c_3+c_4-c_2)P(I+nJ-1) + (c_4-c_2).$$

From condition (8),

$$E(I-1,nJ) - E(I,nJ) > 0 ,$$

we have

$$E(I-1,nJ) - E(I,nJ) = - (c_3+c_4-c_2)P(I+nJ-1) + (c_4-c_2) > 0,$$

or

$$P(I+nJ-1) < \frac{c_4 - c_2}{c_3 + c_4 - c_2} . \qquad (12)$$

Then, from (11) and (12) we have part of the decision rule for the distributor-retailer system,

$$P(I+nJ-1) < \frac{c_4 - c_2}{c_3 + c_4 - c_2} < P(I+nJ). \qquad (13)$$

The right hand side of (9) is

$$E(I,nJ+1) = \sum_{DD=0}^{nJ+1} [c_1(nJ+1-DD) + c_3 I]p(DD)$$

$$+ \sum_{DD=nJ+2}^{I+nJ+1} [c_2(DD-nJ-1) + c_3(I+nJ+1-DD)]p(DD)$$

$$+ \sum_{DD=I+nJ+2}^{\infty} [c_2 I + c_4(DD-I-nJ-1)]p(DD)$$

$$= E(I,nJ) + \sum_{DD=0}^{nJ} c_1 p(DD) + \sum_{DD=nJ+1}^{I+nJ} (c_3-c_2)p(DD)$$

$$- \sum_{DD=I+nJ+1}^{\infty} c_4 p(DD)$$

$$E(I,nJ+1) = E(I,nJ) + c_1 P(nJ) + (c_3-c_2)P(I+nJ)$$

$$- (c_3-c_2)P(nJ) - c_4 + c_4 P(I+nJ)$$

$$= E(I,nJ) + (c_1+c_2-c_3)P(nJ) + (c_4+c_3-c_2)P(I+nJ)-c_4.$$

From condition (9),

$$E(I,nJ+1) - E(I,nJ) > 0,$$

we have

$$E(I,nJ+1) - E(I,nJ) = (c_1+c_2-c_3)P(nJ) + (c_4+c_3-c_2)P(I+nJ) - c_4 > 0,$$

or

$$\frac{c_4}{c_3 + c_4 - c_2} < P(I+nJ) + \frac{c_1 + c_2 - c_3}{c_3 + c_4 - c_2} P(nJ). \qquad (14)$$

The right hand side of (10) is

$$E(I,nJ-1) = \sum_{DD=0}^{nJ-1} [c_1(nJ-1-DD)+c_3 I]p(DD)$$

$$+ \sum_{DD=nJ}^{I+nJ-1} [c_2(DD-nJ+1) + c_3(I+nJ-1-DD)]p(DD)$$

$$+ \sum_{DD=I+nJ}^{\infty} [c_2 I + c_4(DD-I-nJ+1)]p(DD)$$

$$= E(I,nJ) - \sum_{D=0}^{nJ} c_1 p(DD) - c_3 I p(nJ) + c_1 p(nJ)$$

$$+ \sum_{D=nJ+1}^{I+nJ} (c_2-c_3)p(DD) + c_2 p(nJ) + c_3 I p(nJ) - c_3 p(nJ)$$

$$- c_2 I p(I+nJ) - c_2 p(I+nJ) + c_3 p(I+nJ)$$

$$+ \sum_{D=I+nJ+1}^{\infty} c_4 p(DD) + c_2 I p(I+nJ) + c_4 p(I+nJ)$$

$$= E(I,nJ) - c_1 P(nJ) + c_1 p(nJ) + (c_2-c_3)P(I+nJ)$$

$$- (c_2-c_3)P(nJ) + c_2 p(nJ) - c_3 p(nJ) - c_2 p(I+nJ)$$

$$+ c_3 p(I+nJ) + c_4 - c_4 P(I+nJ) + c_4 p(I+nJ)$$

$$= E(I,nJ) - (c_1+c_2-c_3)(P(nJ)-p(nJ))$$

$$+ (c_2-c_3-c_4)(P(I+nJ)-p(I+nJ)) + c_4$$

$$= E(I,nJ) - (c_1+c_2-c_3)P(nJ-1)$$

$$- (c_3+c_4-c_2)P(I+nJ-1) + c_4.$$

From condition (10),

$$E(I,nJ-1) - E(I,nJ) > 0,$$

we have

$$E(I,nJ-1) - E(I,nJ) = -(c_1+c_2-c_3)P(nJ-1)-(c_3+c_4-c_2)P(I+nJ-1) + c_4 > 0,$$

or

$$P(I+nJ-1) + \frac{c_1 + c_2 - c_3}{c_3 + c_4 - c_2} P(nJ-1) < \frac{c_4}{c_3 + c_4 - c_2}. \quad (15)$$

From (14) and (15) we obtain

$$P(I+nJ-1) + \left(\frac{c_1+c_2-c_3}{c_3+c_4-c_2}\right)P(nJ-1) < \frac{c_4}{c_3 + c_4 - c_2} < P(I+nJ) + \left(\frac{c_1+c_2-c_3}{c_3+c_4-c_2}\right)P(nJ). \quad (16)$$

The value of (I,nJ) which satisfies (13) and (16) will be the solution to the system such that the cost to the system is minimized.

In the standard newsboy problem we have only one decision rule, but in this two-echelon model we have two decision rules. The way to solve for I and nJ is that we solve (13) first to get (I+nJ), then put P(I+nJ-1) and P(I+nJ) into

(16) to obtain (nJ). Subtracting (nJ) from (I+nJ) will provide a solution (I,nJ).

The two-echelon inventory model with the decision rules (13) and (16) may be illustrated by an example. Consider a system of one clothing department store and two book stores owned by the same company. The two book stores are located near the bus stations in a big city. Every week the manager of the clothing department store has to decide how many TV Guides to order. He will distribute some to the two book stores and keep some at the clothing department store as a reserve in case the two book stores have a stockout. The clothing department store does not sell the TV Guide itself. According to reliable statistics the manager found out that the demand at each of the two book stores has a Poisson distribution with rate 4 customers per week. The cost of having a TV Guide left over at each book store is 4 cents. If TV Guides are left over at the clothing department store, each one will cost the company 3 cents. In case of stockout at each book store a special immediate delivery will be provided from the clothing department store. This will cost 2 cents per TV Guide. And if the system is out of TV Guides, i.e., demand for them is greater than the total amount at the clothing department store and at the two book stores, this will cost the system 8 cents. The problem is to determine how many TV Guides the manager should order each week, what amount should be kept at a clothing department store, and what amount should be kept at each of the

book stores so that the cost to the system is minimized.

Let

I = Inventory level at the clothing department store,

J = Inventory level at each book store,

D_i = Demand at each book store, $i = 1,2$,

C_1 = Surplus cost per TV Guide at each book store,

C_2 = Cost of special order per each TV Guide from the clothing department store,

C_3 = Surplus cost per TV Guide at the clothing department store,

n = The number of book stores in the system,

and

$$DD = D_1 + D_2 .$$

Then we have:

$$D_i \sim \text{Poisson } (4) , \quad i = 1,2,$$

$$DD = D_1 + D_2 \sim \text{Poisson } (8) ,$$

$$p(D_1) = \frac{e^{-4}(4)^{D_1}}{D_1!} , \quad D_1 = 0,1,2,\ldots,$$

$$p(D_2) = \frac{e^{-4}(4)^{D_2}}{D_2!} , \quad D_2 = 0,1,2,\ldots,$$

and

$$p(DD) = \frac{e^{-8}(8)^{DD}}{DD!} , \quad DD = 0,1,2,\ldots,$$

$$C_1 = 4 , \quad C_2 = 2 , \quad C_3 = 3, \quad C_4 = 8 , \quad n = 2 .$$

From (16) we have

$$P(I+2J-1) + \frac{3}{9} P(2J-1) < \frac{8}{9} < P(I+2J) + \frac{3}{9} P(2J)$$

as the first condition for an optimal solution. From (13) we have

$$P(I+2J-1) < \frac{6}{9} < P(I+2J)$$

as the second condition for an optimal solution.

We wish to find the optimal I and J which satisfy these two conditions, which will be the amount of TV Guides for the system each week.

TABLE I

DISTRIBUTION OF $DD = D_1 + D_2$

$$p(DD) = \frac{.0003355\ (8)^{DD}}{DD!}$$

DD	p(DD)	P(DD)	DD	p(DD)	P(DD)
0	.000335	.000335	9	.124090	.606766
1	.002684	.003019	10	.099272	.706038
2	.010736	.013755	11	.072198	.778236
3	.028629	.042384	12	.048132	.826368
4	.057259	.099643	13	.029620	.855988
5	.091614	.191257	14	.016925	.872913
6	.012215	.203472	15	.009027	.881940
7	.139602	.343074	16	.004513	.886543
8	.139602	.482676	17	.002124	.888667

From Table I we find,

$$P(9) = .606766,$$

and $P(10) = .706038$, so that I+2J=10 satisfies the second condition.

Substituting $P(I+2J-1) = .606766$ and $P(I+2J) = .706038$ into the first condition we have

$$.607 + .333(P(2J-1) < .888 < .706 + .333\ P(2J).$$

The left-hand side becomes

$$.607 + .333\ P(2J-1) < .888$$

or $$P(2J-1) < \frac{.888 - .607}{.333}$$

or $$P(2J-1) < .844 \quad .$$

The right-hand side becomes

$$.706 + .333\ P(2J) > .888$$

or $$P(2J) > \frac{.888 - .706}{.333}$$

or $$P(2J) > .546 \quad .$$

From Table I

$P(9) = .606766$ and $P(8) = .482676$, thus 2J=9 satisfies the first condition.

Since the problem is a discrete case, the TV Guides will be divided unequally between the bookstores.

Since $I+2J = 10$, and $2J = 9$,

then $I = 10 - 9 = 1$,

and the optimum solution to the company is that the manager should order ten TV Guides each week. Nine TV Guides will be at the two book stores, and one TV Guide will be at the clothing department store in case of either one of the two book stores having a stockout. This solution will minimize the total cost to the company concerning selling the TV Guides.

VI. CONCLUSIONS AND EXTENSIONS

In this chapter general conclusions are made concerning the application of the single-period inventory model and the two-echelon single period model presented in Chapter V. Following the conclusions, suggestions are given for extension and enrichment of the model.

The fixed shortage cost case has provided a reasonably useful decision rule. The quadratic cost function seems to provide a difficult decision rule, since I_o can not be solved for readily. This suggests that if possible we might try to solve the problem using linear or piecewise linear cost functions.

The two-echelon single period inventory model has a two-part decision rule, but the optimal solutions may be readily achieved. In a multi-echelon case we will have as many decision rules as the number of the echelons. The two-echelon problem has been rather simplified to facilitate development of decision rules. Examples include identical distributions of demand at the retail level, and no cost interchange among retailers. Nevertheless, the model may provide a useful approximation to some two-echelon real-world problems. Some suggestions for extension and enrichment of the model are listed below.

(1) The problem could be formulated with a non-zero cost for interchange among retailers.

(2) Different demand distributions at the retail level could be used. Solutions might be found by invoking the central limit theorem, so that the total demand to the system could be assumed normal when the number of retailers is large enough.

(3) Several distributors could be included in the system, or a system involving more than two echelons could be structured. Extension and enrichment of the multi-echelon single-period inventory model will add to the model a flexibility and a better representation of real world problems.

It is hoped that the work contained in this paper will be useful to those who are interested in advancing ability to improve inventory systems.

APPENDIX A

DECISION RULE CONSTRUCTION FOR THE QUADRATIC COST FUNCTION

The cost equation of the quadratic cost function is

$$\text{cost} = \begin{cases} c_1(I-D)^2 & , \quad D=0,1,2,\ldots,I \\ c_2(D-I)^2 & , \quad D=I+1,\ldots,D_{max}. \end{cases}$$

(a) Discrete Case

The expected cost for I items, $E(I)$, is

$$E(I) = \sum_{D=0}^{I} c_1(I-D)^2 p(D) + \sum_{D=I+1}^{\infty} c_2(D-I)^2 p(D).$$

It is true that if $E(I)$ is a minimum value then

$$E(I+1) > E(I),$$

and

$$E(I-1) > E(I).$$

These two conditions will be used to construct a decision rule. Replacing I in the $E(I)$ equation by $(I+1)$, we have

$$E(I+1) = \sum_{D=0}^{I+1} c_1(I+1-D)^2 p(D) + \sum_{D=I+2}^{\infty} c_2(D-I-1)^2 p(D),$$

which simplifies to

$$E(I+1) = E(I) + 2\Big(\sum_{D=0}^{I} c_1(I-D)p(D) - \sum_{D=I+1}^{\infty} c_2(D-I)p(D)\Big)$$

$$+ (c_1-c_2)P(I) + c_2.$$

Since $E(I+1) - E(I) > 0$, if $E(I)$ is a minimum,

$$2\Big(\sum_{D=0}^{I} c_1(I-D)p(D) - \sum_{D=I+1}^{\infty} c_2(D-I)p(D)\Big) > -(c_1-c_2)P(I) + c_2. \tag{17}$$

We replace I by (I-1) in the E(I) equation, and since $E(I-1) - E(I) > 0$ if $E(I)$ is a minimum, we obtain

$$2\Big(\sum_{D=0}^{I} c_1(I-D)p(D) - \sum_{D=I+1}^{\infty} c_2(D-I)p(D) < (c_1-c_2)P(I-1) + c_2. \tag{18}$$

From (17) and (18) we obtain the decision rule

$$-(c_1-c_2)P(I_o) + c_2 < 2\Big(\sum_{D=0}^{I_o} c_1(I_o-D)p(D) - \sum_{D=I_o+1}^{\infty} c_2(D-I_o)p(D)\Big)$$

$$< (c_1-c_2)P(I_o-1) + c_2. \tag{3}$$

If we can find the value of I which satisfies this double inequality we will have the optimum choice, I_o.

(b) Continuous Case

The expected cost equation E(I) is

$$E(I) = c_1 \int_{D=0}^{I} (I-D)^2 f(D)\, dD + c_2 \int_{D=I}^{\infty} (D-I)^2 f(D)\, dD.$$

The first derivative, when equated to zero, simplifies to

$$\frac{dE(I)}{dI} = 2I(c_1-c_2)F(I)-2(c_1+c_2) \int_{D=0}^{I} Df(D)dD+2c_2(I-E(D)) = 0.$$

Thus the decision rule for the continuous case becomes

$$F(I_o) = \frac{c_1+c_2}{I_o(c_1-c_2)} \int_{D=0}^{I} Df(D)dD - \frac{c_2}{I_o(c_1-c_2)} (I_o-E(D)), \tag{4}$$

and the expected cost is minimized for the value of I which satisfies this condition.

APPENDIX B

DECISION RULE CONSTRUCTION FOR THE FIXED SHORTAGE COST CASE

The cost equation with fixed shortage cost is

$$\text{cost} = \begin{cases} c_1(I-D) & , \quad D=0,1,2,\ldots,I \\ K & , \quad D=I+1,\ldots,D_{max}. \end{cases}$$

(a) Discrete Case

The expected cost equation is

$$E(I) = \sum_{D=0}^{I} c_1(I-D)p(D) + K \sum_{D=I+1}^{\infty} p(D).$$

Replacing I by I+1, we obtain

$$E(I+1) = E(I) + c_1P(I) - Kp(I+1). \tag{19}$$

Similarly, replacing I by I-1, we obtain

$$E(I-1) = E(I) - c_1P(I-1) + Kp(I). \tag{20}$$

Substituting (19) and (20) into the condition

$$E(I+1) - E(I) > 0,$$

and

$$E(I-1) - E(I) > 0,$$

for a minimum.

We obtain

$$\frac{P(I)}{p(I+1)} > \frac{K}{c_1} \quad ,$$

and

$$\frac{P(I-1)}{p(I)} < \frac{K}{c_1} \quad .$$

Thus the decision rule is

$$\frac{P(I_o-1)}{p(I_o)} < \frac{K}{c_1} < \frac{P(I_o)}{p(I_o+1)} \quad . \tag{5}$$

If we can find the optimum value of I, I_o, which satisfies the double inequality above, then the cost to the system is minimized.

(b) Continuous Case

The expected cost equation is

$$E(I) = c_1 \int_{D=0}^{I} (I-D)f(D)\, dD + K \int_{D=I}^{\infty} f(D)\, dD.$$

Setting the first derivative equal to zero yields

$$\frac{dE(I)}{dI} = c_1If(I) + c_1F(I) - c_1If(I) - Kf(I) = 0,$$

and the decision rule becomes

$$F(I_o) = \frac{K}{c_1} f(I_o).$$

The optimum value of I, I_o, which satisfies the condition above will provide a minimum cost to the system.

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
An Investigation of Optimal Decision Rules for Several Single-Period Stochastic Inventory Problems

5. TYPE OF REPORT & PERIOD COVERED
Master's Thesis; September 1973

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Sirichoke Swasdikiat

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Naval Postgraduate School
Monterey, California 93940

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Naval Postgraduate School
Monterey, California 93940

12. REPORT DATE
September 1973

13. NUMBER OF PAGES
41

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)
Naval Postgraduate School
Monterey, California 93940

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Single-Period Inventory Problems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
The single-period inventory model, known as the newsboy or Christmas tree problem, is extended for several cost expressions and optimal decision rules for these variations are derived. A two-echelon single-period inventory model is developed and optimal decision rules are found. Conclusions and suggested extensions for the newsboy problem and of the two-echelon inventory model are discussed.

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Thesis 146246
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c.1 An investigation of
optimal decision rules
for several single-period
stochastic inventory problems.